wjec cbac

GCE AS/A Level MATHEMATICS

GCE AS/A Level FURTHER MATHEMATICS

FORMULA BOOKLET

From September 2017

CONTENTS

CONTENTS (continued)

Pure Mathematics

Mensuration

Surface area of sphere = $4\pi r^2$

Area of curved surface of cone = $\pi r \times \text{slant height}$

Arithmetic Series

$$S_n = \tfrac{1}{2}n(a+l) = \tfrac{1}{2}n[2a+(n-1)d]$$

Geometric Series

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \qquad \text{for } |r|<1$$

Summations

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2$$

Binomial Series

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n \qquad (n \in \mathbb{N})$$

where $\binom{n}{r} = {}^nC_r = \dfrac{n!}{r!(n-r)!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2} x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r} x^r + \ldots \qquad (|x| < 1,\ n \in \mathbb{R})$$

Logarithms and exponentials

$$e^{x \ln a} = a^x$$

Complex Numbers

$$\{r(\cos\theta + i\sin\theta)\}^n = r^n(\cos n\theta + i\sin n\theta)$$

The roots of $z^n = 1$ are given by $z = e^{\frac{2\pi k i}{n}}$, for $k = 0, 1, 2, \ldots, n-1$

Maclaurin's and Taylor's Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^r}{r!}f^{(r)}(0) + \ldots$$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^r}{r!}f^{(r)}(a) + \ldots$$

Maclaurin's and Taylor's Series

$$f(a+x)=f(a)+xf'(a)+\frac{x^2}{2!}f''(a)+\ldots+\frac{x^r}{r!}f^{(r)}(a)+\ldots$$

$$\mathrm{e}^x=\exp(x)=1+x+\frac{x^2}{2!}+\ldots+\frac{x^r}{r!}+\ldots \qquad \text{for all } x$$

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots+(-1)^{r+1}\frac{x^r}{r}+\ldots \qquad (-1<x\le 1)$$

Maclaurin's and Taylor's Series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots \qquad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots \qquad \text{for all } x$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^r \frac{x^{2r+1}}{2r+1} + \ldots \qquad (-1 \le x \le 1)$$

Maclaurin's and Taylor's Series

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2r+1}}{(2r+1)!} + \ldots \qquad \text{for all } x$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2r}}{(2r)!} + \ldots \qquad \text{for all } x$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2r+1}}{2r+1} + \ldots \qquad (-1 < x < 1)$$

Hyperbolic Functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^{-1} x = \ln\left\{x + \sqrt{x^2 - 1}\right\} \qquad (x \geq 1)$$

$$\sinh^{-1} x = \ln\left\{x + \sqrt{x^2 + 1}\right\}$$

$$\tanh^{-1} x = \tfrac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \qquad (|x| < 1)$$

Trigonometric Identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \left(A \pm B \neq \left(k + \tfrac{1}{2}\right)\pi\right)$$

For $t = \tan \frac{1}{2} A$: $\sin A = \dfrac{2t}{1+t^2} \qquad \cos A = \dfrac{1-t^2}{1+t^2}$

Trigonometric Identities

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

Vectors

The resolute of $\mathbf{a}$ in the direction of $\mathbf{b}$ is $\dfrac{\mathbf{a}.\mathbf{b}}{|\mathbf{b}|}$

The point dividing *AB* in the ratio $\lambda : \mu$ is $\dfrac{\mu\mathbf{a} + \lambda\mathbf{b}}{\lambda + \mu}$

The equation of a plane in Cartesian form is $n_1 x + n_2 y + n_3 z = k$

The perpendicular distance between two skew lines is $D = \dfrac{|(\mathbf{b} - \mathbf{a}).\mathbf{n}|}{|\mathbf{n}|}$ where $\mathbf{a}$ and $\mathbf{b}$ are position vectors of points on each line and $\mathbf{n}$ is a mutual perpendicular to both lines.

Vectors

The perpendicular distance between a point and a line is $D=\frac{|ax_1+by_1-c|}{\sqrt{a^2+b^2}}$ where the coordinates of the point are (x_1, y_1) and the equation of the line is given by $ax+by=c$

The perpendicular distance between a point and a plane is $D=\frac{|n_1\alpha+n_2\beta+n_3\gamma-k|}{\sqrt{n_1^2+n_2^2+n_3^2}}$ where (α,β,γ) are the coordinates of the point and $n_1x+n_2y+n_3z=k$ is the equation of the plane.

Matrix transformations in 2-D

Anticlockwise rotation through θ about O: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Reflection in the line $y = (\tan\theta)x$: $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

Matrix transformations in 3-D

Anticlockwise rotation through θ about:

x-axis	**y-axis**	**z-axis**
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$	$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

where

- an anticlockwise (or positive) rotation about Ox (or x-axis) is in the sense $\mathbf{j} \rightarrow \mathbf{k}$
- an anticlockwise (or positive) rotation about Oy (or y-axis) is in the sense $\mathbf{k} \rightarrow \mathbf{i}$
- an anticlockwise (or positive) rotation about Oz (or z-axis) is in the sense $\mathbf{i} \rightarrow \mathbf{j}$

Differentiation

Function	**Derivative**
$\dfrac{f(x)}{g(x)}$	$\dfrac{f'(x)g(x)-f(x)g'(x)}{\left(g(x)\right)^2}$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

Differentiation

Function	**Derivative**
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$

Differentiation

Function	**Derivative**
$\sinh^{-1} x$	$\dfrac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$\dfrac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\dfrac{1}{1-x^2}$

Integration (+ *constant*; $a > 0$ *where relevant*)

Function	**Integral**
$\tan x$	$\ln\lvert\sec x\rvert$
$\cot x$	$\ln\lvert\sin x\rvert$
$\operatorname{cosec} x$	$-\ln\lvert\operatorname{cosec} x + \cot x\rvert = \ln\left\lvert\tan\left(\tfrac{1}{2}x\right)\right\rvert$
$\sec x$	$\ln\lvert\sec x + \tan x\rvert = \ln\left\lvert\tan\left(\tfrac{1}{2}x + \tfrac{1}{4}\pi\right)\right\rvert$
$\sec^2 x$	$\tan x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ln\cosh x$

Integration (+ *constant*; $a > 0$ *where relevant*)

Function	**Integral**	
$\dfrac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$	$(\lvert x \rvert < a)$
$\dfrac{1}{a^2 + x^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$	
$\dfrac{1}{\sqrt{x^2 - a^2}}$	$\cosh^{-1}\left(\dfrac{x}{a}\right) = \ln\left\{x + \sqrt{x^2 - a^2}\right\}$	$(x > a)$
$\dfrac{1}{\sqrt{a^2 + x^2}}$	$\sinh^{-1}\left(\dfrac{x}{a}\right) = \ln\left\{x + \sqrt{x^2 + a^2}\right\}$	

Integration (+ *constant*; $a > 0$ *where relevant*)

Function	**Integral**	
$\dfrac{1}{a^2 - x^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{a+x}{a-x}\right\rvert = \dfrac{1}{a}\tanh^{-1}\left(\dfrac{x}{a}\right)$	$(\lvert x\rvert < a)$
$\dfrac{1}{x^2 - a^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{x-a}{x+a}\right\rvert$	

Integration (+ *constant*; $a > 0$ *where relevant*)

$$\int u \frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Area of a sector

$$A = \frac{1}{2}\int r^2\,\mathrm{d}\theta \qquad \text{(polar coordinates)}$$

Numerical Mathematics

Numerical integration

The trapezium rule: $\int_a^b y\,\mathrm{d}x \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$ where $h = \frac{b-a}{n}$

Numerical Solution of Equations

The Newton-Raphson iteration for solving $f(x) = 0$: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Mechanics

Motion in a circle

Transverse velocity: $v = r\dot{\theta} = \omega r$

Radial acceleration: $-r\dot{\theta}^2 = -\frac{v^2}{r} = -\omega^2 r$

Centres of Mass of Uniform Bodies

Triangular lamina: $\frac{2}{3}$ along median from vertex

Semi circle: $\frac{4r}{3\pi}$ from straight edge along axis of symmetry

Quarter circle: $\bar{x} = \frac{4r}{3\pi}$ $\bar{y} = \frac{4r}{3\pi}$ from vertex

Probability & Statistics

Probability

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B \mid A)$$

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A)\,\mathrm{P}(B \mid A)}{\mathrm{P}(A)\,\mathrm{P}(B \mid A) + \mathrm{P}(A')\,\mathrm{P}(B \mid A')}$$

Bayes' Theorem: $\mathrm{P}(A_j \mid B) = \dfrac{\mathrm{P}(A_j)\,\mathrm{P}(B \mid A_j)}{\sum \mathrm{P}(A_i)\,\mathrm{P}(B \mid A_i)}$

Discrete distributions

For a discrete random variable X taking values x_i with probabilities p_i

Expectation (mean): $\mathrm{E}(X) = \mu = \sum x_i p_i$

Variance: $\mathrm{Var}(X) = \sigma^2 = \sum (x_i - \mu)^2 p_i = \sum x_i^2 p_i - \mu^2$

For a function $g(X)$: $\mathrm{E}\big(g(X)\big) = \sum g(x_i) p_i$

Standard discrete distributions:

Distribution of X	$\mathrm{P}(X = x)$	Mean	Variance
Binomial $\mathrm{B}(n, p)$	$\binom{n}{x} p^x (1-p)^{n-x}$	np	$np(1-p)$
Poisson $\mathrm{Po}(\lambda)$	$\mathrm{e}^{-\lambda} \frac{\lambda^x}{x!}$	λ	λ

Continuous distributions

For a continuous random variable X having probability density function f

Expectation (mean): $\mathrm{E}(X) = \mu = \int x f(x)\,\mathrm{d}x$

Variance: $\mathrm{Var}(X) = \sigma^2 = \int (x-\mu)^2 f(x)\,\mathrm{d}x = \int x^2 f(x)\,\mathrm{d}x - \mu^2$

For a function $g(X)$: $\mathrm{E}\big(g(X)\big) = \int g(x) f(x)\,\mathrm{d}x$

Cumulative distribution function: $F(x) = \mathrm{P}\big(X \le x\big) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$

Standard continuous distributions:

Distribution of X	P.D.F.	Mean	Variance
Uniform (Rectangular) on $[a,b]$ $\mathrm{U}[a,b]$	$\dfrac{1}{b-a}$	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$
Normal $\mathrm{N}\left(\mu,\sigma^2\right)$	$\dfrac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	μ	σ^2
Exponential $\mathrm{Exp}(\lambda)$	$\lambda\mathrm{e}^{-\lambda x}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$

Expectation algebra

For independent random variables X and Y

$$\mathrm{E}(XY) = \mathrm{E}(X)\,\mathrm{E}(Y)$$

$$\mathrm{Var}(aX \pm bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$$

Sampling distributions

For a random sample $X_1, X_2, \ldots, X_n$ of n independent observations from a distribution having mean μ and variance σ^2

$\bar{X}$ is an unbiased estimator of μ, with $\mathrm{Var}\left(\bar{X}\right) = \dfrac{\sigma^2}{n}$

S^2 is an unbiased estimator of σ^2, where $S^2 = \dfrac{\sum\left(X_i - \bar{X}\right)^2}{n-1}$

For a random sample of n observations from $\mathrm{N}\left(\mu, \sigma^2\right)$

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0,1)$$

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{(n-1)}$$

If X is the observed number of successes in n independent Bernoulli trials in each of which the probability of success is p, and $Y = \frac{X}{n}$ then

$$\mathrm{E}(Y) = p \quad \text{and} \quad \mathrm{Var}(Y) = \frac{p(1-p)}{n}$$

For a random sample of n_x observations from $\mathrm{N}\left(\mu_x, \sigma_x^2\right)$ and, independently, a random sample of n_y observations from $\mathrm{N}\left(\mu_y, \sigma_y^2\right)$

$$\frac{\left(\bar{X} - \bar{Y}\right) - \left(\mu_x - \mu_y\right)}{\sqrt{\frac{\sigma_x^2}{n_x} + \frac{\sigma_y^2}{n_y}}} \sim \mathrm{N}(0,1)$$

Correlation and Regression

For a sample of n pairs of observations (x_i, y_i)

$$S_{xx} = \sum (x_i - \bar{x})^2 = \sum x_i^2 - \frac{\left(\sum x_i\right)^2}{n}$$

$$S_{yy} = \sum (y_i - \bar{y})^2 = \sum y_i^2 - \frac{\left(\sum y_i\right)^2}{n}$$

$$S_{xy} = \sum (x_i - \bar{x})(y_i - \bar{y}) = \sum x_i y_i - \frac{\left(\sum x_i\right)\left(\sum y_i\right)}{n}$$

A measure of linear association between two variables X and Y is given by the Pearson product - moment correlation coefficient r.

For the sample $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, it is given by $r = \dfrac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$

Given data, the parameters α and β of the linear regression model may be estimated using the principle of least squares.

The least squares estimate $\hat{\beta}$ of the parameter β is given by $\hat{\beta} = \dfrac{S_{xy}}{S_{xx}}$

The least squares estimate $\hat{\alpha}$ of the parameter α is given by $\hat{\alpha} = \overline{y} - \hat{\beta}\overline{x}$

The least squares regression line is given by $y = \hat{\alpha} + \hat{\beta}x$

Spearman's rank correlation coefficient is given by $r_s = 1 - \dfrac{6\sum d_i^2}{n\left(n^2 - 1\right)}$