GCE AS/A LEVEL

WJEC GCE AS/A Level in

FURTHER MATHEMATICS

APPROVED BY QUALIFICATIONS WALES

GUIDANCE FOR TEACHING

Teaching from 2017

This Qualifications Wales regulated qualification is not available to centres in England.

CONTENTS

INTRODUCTION

The **WJEC AS and A Level Further Mathematics** qualifications, approved by Qualification Wales for first teaching from September 2017, are available to:

- All schools and colleges in Wales
- Schools and colleges in independent regions such as Northern Ireland, Isle of Man and the Channel Islands

The AS will be awarded for the first time in Summer 2018, using grades A – E; the A level will be awarded for the first time in Summer 2019, using grades A* – E.

The qualification provides a broad, coherent, satisfying and worthwhile course of study. It encourages learners to develop confidence in, and a positive attitude towards, mathematics and to recognise its importance in their own lives and to society.

The specification builds on the tradition and reputation WJEC has established for clear, reliable assessment supported by straightforward, accessible guidance and administration.

In addition to this guide, support is provided in the following ways:

- Specimen assessment materials
- Face-to-face CPD events
- Examiners' reports on each question paper
- Free access to past question papers and mark schemes via the secure website
- Direct access to the subject officer
- Free online resources
- Exam Results Analysis
- Online Examination Review

AIMS OF THE GUIDANCE FOR TEACHING

The principal aims of this Guidance for Teaching are to offer support to teachers in delivery of the new WJEC GCE AS/A Level Further Mathematics specification and to offer guidance on the requirements of the qualification and the assessment process.

The guide is **not intended as a comprehensive reference**, but as support for professional teachers to develop stimulating and exciting courses tailored to the needs and skills of their own students in their particular institutions.

The guide contains detailed clarification and guidance on the subject content for all units in the qualification.

The guide also contains a section on assessment objectives and how the different elements of these can be assessed in examination papers.

Written examination: 1 hours 30 minutes
$13\frac{1}{3}$% of A level qualification ($33\frac{1}{3}$% of AS qualification)
70 marks

Topics	Guidance
2.1.1 Proof	
Construct proofs using mathematical induction. Contexts include sums of series, powers of matrices and divisibility.	Including application to the proof of the binomial theorem for a positive integral power. eg. the proof of the divisibility of $5^{2n} - 1$ by 24. *Knowledge of the Σ notation is assumed.*
2.1.2 Complex Numbers	
Solve any quadratic equation with real coefficients. Solve cubic or quartic equations with real coefficients (given sufficient information to deduce at least one root for cubics or at least one complex root or quadratic factor for quartics).	
Add, subtract, multiply and divide complex numbers in the form $x + \mathrm{i}y$, with x and y real. Understand and use the terms 'real part' and 'imaginary part'.	Be familiar with 'modulus' and 'argument'.

Topics	Guidance
Understand and use the complex conjugate. Know that non-real roots of polynomial equations with real coefficients occur in conjugate pairs	The complex conjugate of z will be denoted by $\overline{z}$.
Equate the real and imaginary parts of a complex number.	Including the solution of equations such as $z+2\overline{z}=\frac{1+2\mathrm{i}}{1-\mathrm{i}}$.
Use and interpret Argand diagrams	Includes representing complex numbers by points in an Argand diagram.
Understand and use the Cartesian (algebraic) and modulus-argument (trigonometric) forms of a complex number. Convert between the Cartesian form and modulus-argument form of a complex number.	$z = x + \mathrm{i}y$ and $z = r(\cos\theta + \mathrm{i}\sin\theta)$ where $\theta = \arg(z)$ may be taken to be in either $[0, 2\pi)$ or $(-\pi, \pi]$ or $[0, 360^{\circ})$ or $(-180^{\circ}, 180^{\circ}]$. *Knowledge of radians is assumed.*
Multiply and divide complex numbers in modulus-argument form.	*Knowledge of radians and compound angle formulae is assumed.*
Construct and interpret simple loci in an Argand diagram, such as $\lvert z-a\rvert > r$ and $\arg(z-a)=\theta$.	For example, $\lvert z-1\rvert = 2\lvert z+\mathrm{i}\rvert$. *Knowledge of radians is assumed.*
Simple cases of transformations of lines and curves defined by $w = f(z)$.	For example, the image of the line $x + y = 1$ under the transformation defined by $w = z^2$.

Topics	Guidance
2.1.3 Matrices	
Add, subtract and multiply conformable matrices. Multiply a matrix by a scalar.	
Understand and use zero and identity matrices. Understand and use the transpose of a 2 x 2 matrix.	
Use matrices to represent • linear and non-linear transformations in 2-D, involving 2 x 2 and 3 x 3 matrices, • successive transformations, • single transformations in 3-D (3-D transformations confined to reflection in one of x = 0, y = 0, z = 0 or rotation about one of the coordinate axes).	Transformations to only include translation, rotation and reflection, using 2 x 2 and/or 3 x 3 matrices. Knowledge that the transformation represented by **AB** is the transformation represented by **B** followed by the transformation represented by **A**. *Knowledge of 3-D vectors is assumed.*
Find invariant points and lines for linear and non-linear transformations.	
Calculate determinants of 2 x 2 matrices.	Use and understand the notation $\lvert\mathbf{M}\rvert$ or det **M** or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ or Δ. Determinant as an area scale factor in transformations.
Understand and use singular and non-singular matrices. Understand and use properties of inverse matrices. Calculate and use the inverse of non-singular 2 x 2 matrices.	

Topics	Guidance
2.1.4 Further Algebra and Functions	
Understand and use the relationship between roots and coefficients of polynomial equations up to quartic equations.	
Form a polynomial equation whose roots are a transformation or a combination of the roots of a given polynomial equation.	Roots of the new equation to include squares, cubes, reciprocals of the roots of the original equation. Roots could be in arithmetic or geometric progression. *Knowledge of arithmetic and geometric progressions is assumed.*
Understand and use formulae for the sums of integers, squares and cubes and use these to sum other series. Understand and use the method of differences for summation of series, including the use of partial fractions.	Summation of a finite series. Use of formulae for $\sum_{r=1}^{n} r$, $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r^3$. Including mathematical induction (see section on Proof) and difference methods. Summation of series such as $\sum_{r=1}^{n} \frac{1}{r(r+1)}$ and $\sum_{r=1}^{n} (2r+1)^3$. *Knowledge of the Σ notation and partial fractions is assumed.*

Topics	Guidance
2.1.5 Further Vectors	
Understand and use the vector and Cartesian forms of an equation of a straight line in 3-D.	$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and $\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$ *Knowledge of 3-D vectors is assumed.*
Understand and use the vector and Cartesian forms of the equation of a plane.	Vector equation of a plane: $(\mathbf{r} - \mathbf{r}_0).\mathbf{n} = 0$. Cartesian equation of a plane: $n_1x + n_2y + n_3z = k$.
Calculate the scalar product and use it to express the equation of a plane, and to calculate the angle between two lines, the angle between two planes and the angle between a line and a plane.	$\mathbf{a}.\mathbf{b} = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta$ The form $\mathbf{r}.\mathbf{n} = k$ for a plane.
Use the scalar product to check whether vectors are perpendicular.	If two vectors are perpendicular, then $\mathbf{a}.\mathbf{b} = 0$.
Find the intersection of a line and a plane.	
Calculate the perpendicular distance between two lines, from a point to a line and a point to a plane.	The perpendicular distance between two lines, from a point to a line and a point to a plane is the shortest distance between them.

Written examination: 1 hours 30 minutes
$13\frac{1}{3}$% of A level qualification ($33\frac{1}{3}$% of AS qualification)
70 marks

Topics	Guidance
2.2.1 Random Variables and the Poisson Process	
Understand and use the mean and variance of linear combinations of independent random variables. ie. use of results: $E(aX + b) = aE(X) + b$ $Var(aX + b) = a^2Var(X)$ $E(aX + bY) = aE(X) + bE(Y)$ For independent X and Y, use the results: $E(XY) = E(X)\,E(Y)$ $Var(aX + bY) = a^2Var(X) + b^2Var(Y)$	For discrete and continuous random variables. To include the binomial and Poisson distributions, ie. If $X \sim B(n, p)$ then $E(X) = np$ and $Var(X) = np(1-p)$. If $Y \sim Po(\lambda)$ then $E(Y) = \lambda$ and $Var(Y) = \lambda$.
Probability: Discrete probability distributions. Find the mean and variance of simple discrete probability distributions.	Use of $E(X) = \mu = \sum x P(X = x)$ and $Var(X) = \sigma^2 = \sum x^2 P(X = x) - \mu^2$

Topics	Guidance
Probability: Continuous probability distributions.	
Understand and use probability density and cumulative distribution functions and their relationships.	Use of the results $f(x) = F'(x)$ and $F(x) = \int_{-\infty}^{x} f(t)\,dt$.
Find and use the median, quartiles and percentiles.	eg. $F(m) = 0.5$, where m is the median, $F(u) = 0.75$, where u is the upper quartile.
Find and use the mean, variance and standard deviation.	
Understand and use the expected value of a function of a continuous random variable.	E[$g(X)$] = $\int g(x)\,f(x)\,dx$ Simple functions only, e.g. $\frac{1}{X^2}$ and $\sqrt{X}$.

Topics	Guidance
Statistical distributions: Poisson and exponential distributions.	Use of formula and tables/calculator for Poisson distribution.
Find and use the mean and variance of a Poisson distribution and an exponential distribution.	Knowledge and use of: If $X \sim \text{Po}(\lambda)$ then $\text{E}(X) = \lambda$ and $\text{Var}(X) = \lambda$ If $Y \sim \text{Exp}(\lambda)$ then $\text{E}(Y) = \frac{1}{\lambda}$ and $\text{Var}(Y) = \frac{1}{\lambda^2}$
Understand and use Poisson as an approximation to the binomial distribution.	Knowledge and use of: If $X \sim \text{B}(n, p)$ then $E(X) = np$.
Apply the result that the sum of independent Poisson random variables has a Poisson distribution.	
Use of the exponential distribution as a model for intervals between events.	Learners will be expected to know that $\frac{\text{d}}{\text{d}x}\left(\text{e}^{kx}\right) = k\text{e}^{kx}$

Topics	Guidance
2.2.2 Exploring relationships between variables and goodness of fit of a model	
Understand and use correlation and linear regression: Explore the relationships between several variables. Calculate and interpret • Spearman's rank correlation coefficient • Pearson's product-moment correlation coefficient. Calculate and interpret the coefficients for a least squares regression line in context; interpolation and extrapolation.	To include tests for significance. Excludes tied ranks. Use of tables for Spearman's and Pearson's product moment correlation coefficients. Be able to choose between Spearman's rank correlation coefficient and Pearson's product-moment correlation coefficient for a given context. Including from summary statistics.
Understand and use the Chi-squared distribution: Conduct goodness of fit test using $\sum\frac{(O-E)^2}{E}$, or equivalent form, as an approximate χ^2 statistic (for use with categorical data). Use χ^2 test to test for association in a contingency table and interpret results	For use with binomial, discrete uniform and Poisson distributions, for known parameters only. To include pooling. Not including Yates continuity correction.

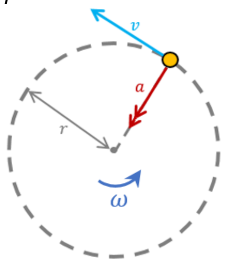
Written examination: 1 hours 30 minutes
$13\frac{1}{3}$% of A level qualification ($33\frac{1}{3}$% of AS qualification)
70 marks

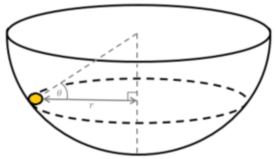
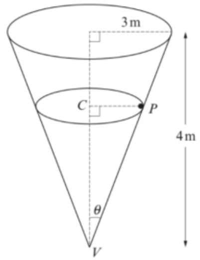
Topics	Guidance
2.3.1 Momentum and Impulse	
Understand and use momentum and impulse. Understand and use conservation of momentum. Understand and use Newton's Experimental Law for (i) the direct impact of two bodies moving in the same straight line, (ii) the impact of a body moving at right-angles to a plane.	Problems will be restricted to the one-dimensional case. Learners should be able to recall, understand and use • the definition of impulse, $I = Ft$ (for a constant force); • the fact that, when two bodies collide, the impulse acting on each body is equal and opposite (Newton's Third Law); • the definition of momentum, $\text{momentum} = mv$; • the impulse-momentum principle, $I = mv - mu$ (change in momentum); • the principle of conservation of momentum, $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$. To include impact with planes. For example, • a body moving horizontally and impacting with a wall, • a ball falling vertically (under gravity) and bouncing off the ground. Problems may involve successive collisions and/or the necessary conditions for further collisions to occur. One body may be stationary. . Newton second (N s) is the unit for both momentum and impulse. Learners should be able to understand and use the concept of the

Topics	Guidance
	coefficient of restitution, e. In particular, • e is dimensionless, • $0 \leq e \leq 1$, • $e = 0$, for inelastic collisions where bodies will coalesce (maximum loss in KE), • $e = 1$, for perfectly elastic collisions (no loss in KE), Newton's Experimental Law: $$e = \frac{\text{speed of separation}}{\text{speed of approach}}$$ Application to specific cases: (i) Impact between two particles, $$v_2 - v_1 = -e(u_2 - u_1)$$ (ii) Impact with a plane, $$v = -eu$$ Learners should be able to make interconnections with the energy concepts covered in 2.3.2 below. For example, working out the percentage loss in KE due to a collision.

Topics	Guidance
2.3.2 Hooke's Law, Work, Energy and Power	
Solve problems involving light strings and springs obeying Hooke's Law.	Learners should be able to understand and use Hooke's Law: *The tension in a stretched elastic string/spring is proportional to its extension, i.e.* $T = kx$*, where* k *is the constant of proportionality.* Learners should be able to understand and use the concept of the modulus of elasticity, λ (measured in newtons, N), and therefore use Hooke's Law in the form, $$\mathrm{T} = \frac{\lambda x}{l}$$ Application of the above for compressed springs.
Understand and use work, energy and power. Understand and use gravitational potential energy, kinetic energy, elastic energy.	Learners should be able to understand and use the concept of • the work done by (or against) a force (measured in joules, J); • energy as the capacity to do work (measured in joules, J); • power as the rate at which a force does work, i.e. $P = \frac{\text{work done}}{\text{total time}}$ (measured in watts, W). Learners should be able to recall and use the following formulae: • $W = Fd$, for a constant force, • $\mathrm{PE} = mgh$, • $\mathrm{KE} = \frac{1}{2}mv^2 \left(\text{or } \frac{1}{2}m\,\mathbf{v}.\mathbf{v}\right)$, • $\mathrm{EE} = \frac{\lambda x^2}{2l}$ • $P = Fv$ $(\text{or } \mathbf{P} = \mathbf{F}.\mathbf{v})$, for power at an instant.

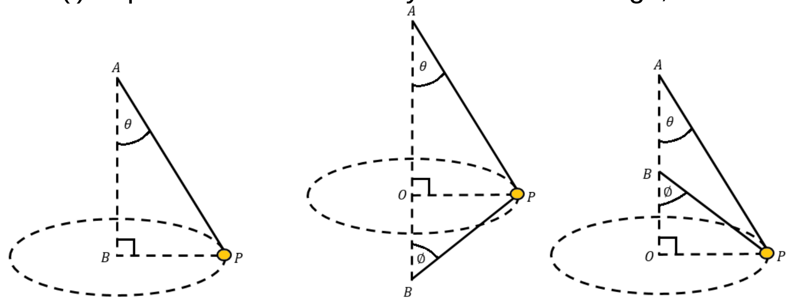
Topics	Guidance
Understand and use conservation of energy. Understand and use the Work-energy Principle.	Learners should be able to understand and use the concept of • mechanical energy of a body (sum of PE and KE), • mechanical energy of a system including elastic strings/springs (sum of PE, KE, and EE). Conservation of energy: *The total **energy** of an isolated system remains constant.* Learners should be able to understand and use the fact that the total energy of a system maybe increased (or decreased) by work done by (or against) an external force. For example, • energy decrease due to work done by friction, eg. a mass sliding down a rough inclined plane; • energy decrease due to work done by air resistance, eg. a large object falling under gravity; • energy increase due to work done, eg. a cyclist climbing a hill. Work-energy principle: *The total work done by all forces acting on a body equals the change in the kinetic energy of the body.* Equivalently, work done = change in energy.

Topics	Guidance
2.3.3 Circular Motion	
Understand and use circular motion.	Learners should be able to recall, understand and use • the link between speed, v, and angular speed, ω, i.e. $v = r\omega$; • the formulae for radial acceleration in circular motion, i.e. $a = r\omega^2$ and $a = \frac{v^2}{r}$, together with $F = ma$.  For modelling purposes, non-standard units may be used, e.g. revolutions per minute.
Understand and use the motion of a particle in a horizontal circle with uniform angular speed.	Learners should be able to solve problems related to • a particle that is constrained by one string and a smooth horizontal surface. • banked tracks, including the condition for no side slip; • a particle moving on the inner surface of a hollow cone or sphere;

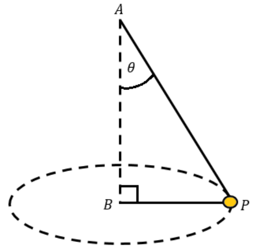
Topics	Guidance
	 

- the conical pendulum, in the following cases;

  (i) a particle constrained by one or two strings,



  (ii) a particle threaded on one string,



Topics	Guidance
	Note that if a particle is **fixed** at some point on a single length of string other than the end point, then it can be treated as a problem with two strings. In particular, the tension either side of the fixed particle may be different. Strings may be elastic. *Knowledge of resolution of forces in any given direction is assumed.*
Understand and use the motion in a vertical circle.	Learners should be able to use conservation of energy to solve problems related to particles • moving on the inside/outside surface of a sphere, • attached to the end of a light inextensible string or light rod. Questions may be set involving full circles or part circles. Learners should be able to determine and use the • condition for a particle to move in complete vertical circles, • points where the circular motion breaks down (e.g. loss of contact with a surface or a string becoming slack). The tangential component of the acceleration is **not** required. *Knowledge of resolution of forces in any given direction is assumed.*

Topics	Guidance
2.3.4 Differentiation and Integration of Vectors	
Differentiate and integrate vectors in component form with respect to a scalar variable. Understand and use vector quantities including displacement, velocity, acceleration, force and momentum.	Learners should be able to recall and use the following for vectors in 3 dimensions; $\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}, \quad \mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}, \quad \mathbf{r} = \int \mathbf{v}\,\mathrm{d}t, \quad \mathbf{v} = \int \mathbf{a}\,\mathrm{d}t,$ where $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{a}$ are given in terms of t. Examples of vectors in component form; • $\mathbf{r} = e^t\mathbf{i} + 4e^{-4t}\mathbf{j} - \cos(t)\mathbf{k}$ • $\mathbf{v} = \sin(3t)\mathbf{i} - 2\cos(5t)\mathbf{j} + 3t^3\mathbf{k}$ • $\mathbf{a} = \sqrt{t}\,\mathbf{i} + 5(t^2 - 1)\mathbf{k}$ Resultants of vector quantities. Learners should be able to solve simple problems including the relative motion of two objects and the determination of the shortest distance between them. *Knowledge of 3-D vectors is assumed.* *Knowledge of differentiation and integration of* e^{kx}, $\sin kx$ *and* $\cos kx$ *is assumed.*

Notes on modelling
Learners should be able to • state any necessary modelling assumptions. For example, - objects/bodies can be modelled as particles (including for problems involving impact), - strings and springs will be light, - strings will be inextensible unless otherwise stated, - 'rough' ('smooth') implies that friction must be considered (ignored), - no external forces are present unless otherwise stated, e.g. air resistance can be ignored, - acceleration due to gravity, g, will be assumed to be constant, irrespective of height and vertical distance travelled. • understand that mechanical energy is lost when possible external forces are considered; • reflect on the significance of the value of e or a range of values of e; • identify possible limitations of any modelling assumptions made; • briefly describe the mathematical consequences if certain modelling requirements are not met; • suggest possible refinements to a chosen mathematical model. Attempts should be made to relate questions to real-life settings wherever possible. For example; sporting situations, bungee jumping, fairground rides and transport.

Written examination: 2 hours 30 minutes
35% of A level qualification
120 marks

This unit is **compulsory**.

Topics	Guidance
2.4.1 Complex Numbers	
Understand de Moivre's theorem and use it to find multiple angle formulae and sums of series.	To include proof by induction of de Moivre's Theorem for positive integer values of n. For example, showing that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ and $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$.
Know and use the definition $e^{i\theta} = \cos\theta + i\sin\theta$ and the form $z = re^{i\theta}$.	Be able to use $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $i\sin\theta = \frac{1}{2}(e^{i\theta} - e^{-i\theta})$.
Find the n distinct nth roots for $re^{i\theta}$ for $r \neq 0$ and know that they form the vertices of a regular n-gon in the Argand diagram.	
Use complex roots of unity to solve geometric problems.	

Topics	Guidance
2.4.2 Further Trigonometry	
Solve trigonometric equations. Use the formulae for $\sin A \pm \sin B$, $\cos A \pm \cos B$ and for $\sin x$, $\cos x$ and $\tan x$ in terms of t, where $t = \tan \frac{1}{2}x$. Find the general solution of trigonometric equations.	Questions aimed solely at proving identities will not be set. For example, $\cos\theta + \cos 2\theta + \cos 3\theta = 0$ and $2\sin x - \tan \frac{1}{2}x = 0$.
2.4.3 Matrices	
Calculate determinants up to 3 x 3 matrices and interpret as scale factors, including the effect on orientation.	
Calculate and use the inverse of non-singular 3 x 3 matrices.	Including knowledge of the term adjugate matrix.
Solve three linear simultaneous equations in three variables by use of the inverse matrix and by reduction to echelon form. Understand and use the determinantal condition for the solution of simultaneous equations which have a unique solution.	To include equations which (a) have a unique solution, (b) have non-unique solutions, (c) are not consistent.
Interpret geometrically the solution and failure of three simultaneous linear equations.	

Topics	Guidance
2.4.4 Further Algebra and Functions	
Find the Maclaurin series of a function (including the general term)	
Recognise and use the Maclaurin series for e^x, $\ln(1+x)$, $\sin x$, $\cos x$ and $(1+x)^n$, and be aware of the range of values of x for which they are valid.	Proof not required.
Understand and use partial fractions with denominators of the form $(ax+b)(cx^2+d)$.	To include improper fractions.
2.4.5 Further Calculus	
Evaluate improper integrals, where either the integrand is undefined at a value in the range of integration or the range of integration extends to infinity.	
Derive formulae for and calculate volumes of revolution.	Rotation may be about the x-axis or the y-axis. Equations may be in Cartesian or parametric form.
Understand and evaluate the mean value of a function.	Mean value of a function $f(x) = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$
Integrate using partial fractions (extend to quadratic factors (ax^2+c) in the denominator).	
Differentiate inverse trigonometric functions.	

Topics	Guidance
Integrate functions of the form $\frac{1}{\sqrt{a^2-x^2}}$ and $\frac{1}{a^2+x^2}$ and be able to choose trigonometric substitutions to integrate associated functions.	
2.4.6 Polar Coordinates	
Understand and use polar coordinates and be able to convert between polar and Cartesian coordinates.	Where $r \geq 0$ and the value of θ may be taken to be in either $[0, 2\pi)$ or $(-\pi, \pi]$.
Sketch curves with r given as a function of θ, including the use of trigonometric functions.	Candidates will be expected to sketch simple curves such as $r = a(b + c\cos\theta)$ and $r = a\cos n\theta$. Includes the location of points at which tangents are parallel to, or perpendicular to, the initial line.
Find the area enclosed by a polar curve.	Excludes the intersection of curves.
2.4.7 Hyperbolic functions	
Understand the definitions of hyperbolic functions, $\sinh x$, $\cosh x$ and $\tanh x$, including their domains and ranges, and be able to sketch their graphs.	$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$, $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$ $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(e^x - e^{-x}\right)}{\left(e^x + e^{-x}\right)}$ Know and use the formulae for $\sinh(A \pm B)$, $\cosh(A \pm B)$, $\tanh(A \pm B)$, $\sinh 2A$, $\cosh 2A$ and $\tanh 2A$. Knowledge and use of the identity $\cosh^2 A - \sinh^2 A \equiv 1$ and its equivalents.

Topics	Guidance
Differentiate and integrate hyperbolic functions.	eg. Differentiate $\sinh 2x$, $x\cosh^2 x$
Understand and be able to use the definitions of the inverse hyperbolic functions and their domains and ranges.	$\sinh^{-1} x = \ln\left[x+\sqrt{x^2+1}\right]$ $\cosh^{-1} x = \ln\left[x+\sqrt{x^2-1}\right], \quad x \ge 1$ $\tanh^{-1} x = \frac{1}{2}\ln\left[\frac{1+x}{1-x}\right], \quad -1 < x < 1$
Derive and use the logarithmic forms of the inverse hyperbolic functions.	
Integrate functions of the form $\frac{1}{\sqrt{x^2+a^2}}$ and $\frac{1}{\sqrt{x^2-a^2}}$, and be able to choose substitutions to integrate associated functions.	
2.4.8 Differential equations	
Find and use an integrating factor to solve differential equations of the form $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$ and recognise when it is appropriate to do so.	
Find both general and particular solutions to differential equations.	
Use differential equations in modelling in a variety of contexts.	Contexts will **not** include mechanics contexts.
Solve differential equations of the form $y'' + ay' + by = 0$, where a and b are constants, by using the auxiliary equation.	

Topics	Guidance
Solve differential equations of the form $y'' + ay' + by = f(x)$, where a and b are constants, by solving the homogenous case and adding a particular integral to the complementary function (in cases where $f(x)$ is a polynomial, exponential or trigonometric function).	$f(x)$ will have one of the forms $A + Bx$, $cx^2 + dx + e$, ke^{qx} or $m\cos\omega x + n\sin\omega x$.
Understand and use the relationship between the cases when the discriminant of the auxiliary equation is positive, zero and negative and the form of solution of the differential equation.	
Analyse and interpret models of situations with one independent variable and two dependent variables as a pair of coupled 1st order simultaneous equations and be able to solve them.	For example, predator-prey models. Restricted to first order differential equations of the form $\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by + f(t)$ $\frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy + g(t)$

Written examination: 1 hours 45minutes
25% of A level qualification
80 marks

Candidates will choose **either** Unit 5 **or** Unit 6.

Topics	Guidance
2.5.1 Samples and Populations	
Understand and use unbiased estimators: Understand and use the variance criterion for choosing between unbiased estimators. Understand and use unbiased estimators of a probability and of a population mean and their standard errors. Understand and use an unbiased estimator of a population variance.	Use of $s^2 = \frac{1}{n-1}\sum(x_i - \overline{x})^2$.

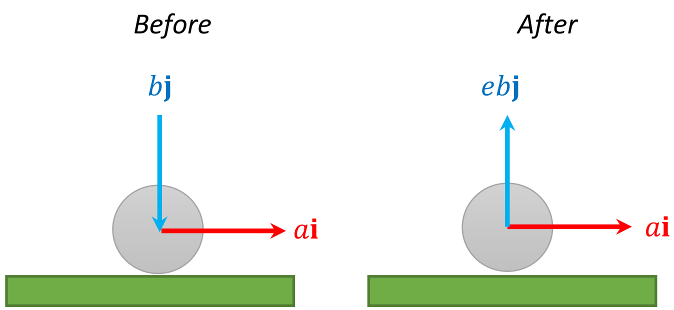
Topics	Guidance
2.5.2 Statistical Distributions	
Understand and use the result that a linear combination of independent normally distributed random variables has a normal distribution. Understand and use the fact that the distribution of the mean of a random sample from a normal distribution with known mean and variance is also normal. Know and use the Central Limit Theorem: Understand and use the fact that the distribution of the mean of a large random sample from any distribution with known mean and variance is approximately normally distributed.	For a population with mean μ and variance σ^2, for large n $\bar{X} \approx \sim N\left(\mu, \frac{\sigma^2}{n}\right)$
2.5.3 Hypothesis Testing	
Understand and use tests for: (a) a specified mean of any distribution whose variance is estimated from a large sample. (b) difference of two means for two independent normal distributions with known variances. (c) a specified mean of a normal distribution with unknown variance. Interpret results for these tests in context.	Using the Central Limit Theorem. The specified difference may be different from zero. To include estimating the variance from the data and using the Student's *t*-distribution. The significance level will be given and questions involving the Student's *t*-distribution will not require the calculation of *p*-values.

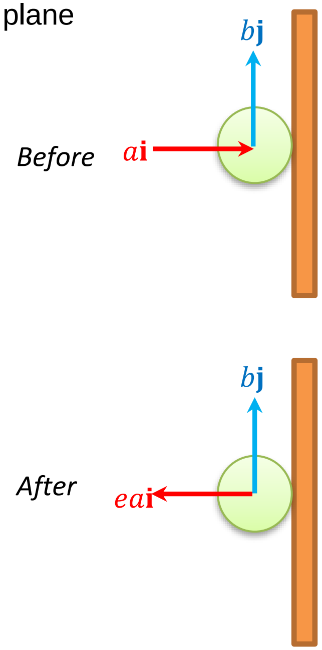
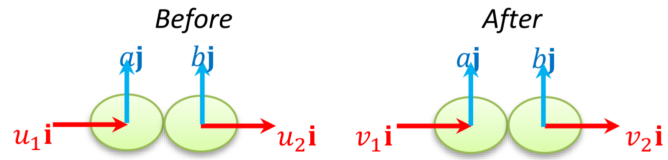
Topics	Guidance
Non-parametric tests: Understand and use Mann-Whitney and Wilcoxon signed-rank tests, understanding appropriate test selection and interpreting the results in context.	Alternative tests for when a distributional model cannot be assumed. Excludes tied ranks. The Wilcoxon signed rank test to include paired samples.
2.5.4 Estimation	
Understand and use confidence intervals: Understand and use confidence limits for (a) the mean of a normal distribution with (i) known variance and (ii) unknown variance, (b) the difference between the means of two normal distributions whose variances are known. Understand and use approximate confidence limits, given large samples, for a probability or a proportion. Interpret results in practical contexts.	Candidates will be expected to be familiar with the term 'confidence interval', including its interpretation. Estimating the variance from the data and using the Student's *t*-distribution. Using a normal approximation.

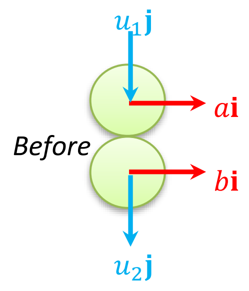
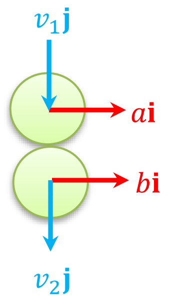
Written examination: 1 hours 45 minutes
25% of A level qualification
80 marks

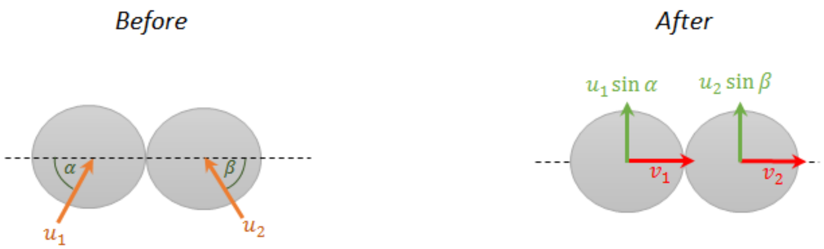
Candidates will choose **either** Unit 5 **or** Unit 6.

Topics	Guidance
2.6.1 Rectilinear motion	
Form and solve simple equations of motion in which (i) acceleration is given as a function of time, displacement or velocity, (ii) velocity is given as a function of time or displacement.	To include use of $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}v}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}x}$.
2.6.2 Momentum and Impulse	
Understand and use momentum and impulse in two dimensions, using vectors.	Extension of section 2.3.1 of AS Unit 3 to include 2D vectors. Oblique impact of spheres and/or a sphere with a horizontal or vertical surface. Multiple collisions may be encountered. Spheres modelled as particles. Learners should be able to recall, understand and use the • impulse-momentum principle; $\mathbf{I} = m\mathbf{v} - m\mathbf{u}$ (change in momentum); • principle of conservation of momentum, $m_1\mathbf{u_1} + m_2\mathbf{u_2} = m_1\mathbf{v_1} + m_2\mathbf{v_2}$.

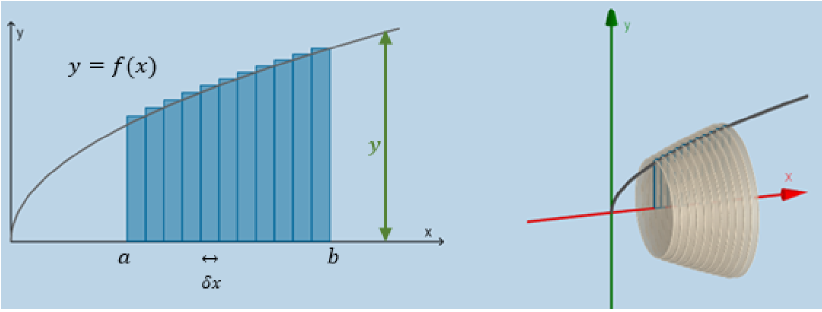
Topics	Guidance
	Learners should: • understand the term oblique impact; • know that the law of restitution can only be applied along the line of impulses, • be able to calculate loss of kinetic energy due to impact During collisions, if restitution calculations are needed, the line joining the centres of the spheres will be parallel to either **i** or **j**. Impact with a plane: • Horizontal plane 

Topics	Guidance
	• Vertical plane  Impact of two spheres: • Horizontally  $v_2 - v_1 = -e(u_2 - u_1)$

Topics	Guidance
	• Vertically   $$v_2 - v_1 = -e(u_2 - u_1)$$ Candidates should be encouraged to draw clear diagrams where necessary. Questions need not be restricted to vectors in the form $a\mathbf{i} + b\mathbf{j}$. However, during collisions, the line joining the centres of the spheres (line of impact) will be horizontal or vertical.

Topics	Guidance
	 $v_2 - v_1 = -e(-u_2 \cos \beta - u_1 \cos \alpha)$

Topics	Guidance
2.6.3 Moments and Centre of Mass	
Understand and use the centre of mass of a coplanar system of particles.	Learners should: • be familiar with the terms 'centre of mass' and 'centre of gravity', • know that the centre of mass of a system of coplanar particles is given by $(\bar{x}, \bar{y})$ where $\bar{x} = \frac{\sum m_i x_i}{\sum m_i}$ and $\bar{y} = \frac{\sum m_i y_i}{\sum m_i}$ (or vector equivalent $\bar{\mathbf{r}} = \frac{\sum m_i \mathbf{r_i}}{\sum m_i}$, where $\mathbf{r}_i = x_i\mathbf{i} + y_i\mathbf{j}$).
Understand and use the centre of mass of uniform laminae: triangles, rectangles, circles, semicircles, quarter-circles and composite shapes.	Learners should understand the terms 'uniform lamina' and 'light rod'. The following results may be quoted without proof. They may be found on page 8 of the Formula Booklet. *Centres of Mass of Uniform Bodies* Triangular lamina: $\frac{2}{3}$ along median from vertex Semi circle: $\frac{4r}{3\pi}$ from straight edge along axis of symmetry Quarter circle: $\bar{x} = \frac{4r}{3\pi}$ $\bar{y} = \frac{4r}{3\pi}$ from vertex Questions may be set in which composite shapes are formed from laminae of different densities (mass per unit area).

Topics	Guidance
Solve problems involving simple cases of equilibrium of a plane lamina and/or a coplanar system of particles connected by light rods.	The lamina or system of particles may be suspended from a fixed point.
Understand and use the centre of mass of uniform rigid bodies and composite bodies.	The use of symmetry and/or integration to determine the centre of mass of a uniform body. Learners should know and be able to use of the results for a solid of revolution.  https://www.geogebra.org/m/hhRJQyz9 Let ρ be the weight per unit volume of the solid and let V be its total volume. Dividing the solid into 'thin' disks, of radius y and thickness δx we can deduce the following. **Body** / **Weight** / **x coordinate of COM** Individual disk / $\rho(\pi y^2 \delta x)$ / x Whole solid / $\rho \sum \pi y^2 \delta x = \rho V$ / $\bar{x}$ Taking moments about the y-axis, we have

Topics	Guidance
	$$\sum_{x=a}^{x=b} \rho(\pi y^2 x\, \delta x) \approx \rho V\bar{x}$$ As $\delta x \to 0$, we obtain the following result for a solid rotated about the x-axis, $$V\bar{x} = \pi \int_a^b xy^2 \mathrm{d}x,$$ where V can be evaluated using a known formula or $V = \pi \int_a^b y^2 \mathrm{d}x$. Similarly, for a solid rotated about the y-axis, $$V\bar{y} = \pi \int_a^b x^2 y \mathrm{d}x,$$ where V can be evaluated using a known formula or $V = \pi \int_a^b x^2 \mathrm{d}y$. Candidates may simply state and use the integral results above. The centres of mass of some common bodies are given in the table below. Proofs will only be requested for **solid** bodies.

Topics	Guidance
	See table below

Diagram	Body	Volume/Curved Surface Area	Height of COM above base
	**Solid** sphere, radius r	$\frac{4}{3}\pi r^3$	r
	**Solid** hemisphere, radius r **Hollow** hemisphere*, radius r	$\frac{2}{3}\pi r^3$ $2\pi r^2$	$\frac{3}{8}r$ $\frac{1}{2}r$
	**Solid** cone or pyramid, height h, radius r **Hollow** cone or pyramid*, height h, radius r	$\frac{1}{3}\pi r^2 h$ πrl *(l is sloping height)*	$\frac{1}{4}h$ $\frac{1}{3}h$

** Not including circular base*

Questions may be set in which the individual parts of a composite rigid body have different densities (mass per unit area/volume).

Topics	Guidance
2.6.4 Equilibrium of Rigid Bodies	
Understand and use the equilibrium of a single rigid body under the action of coplanar forces where the forces are not all parallel.	Problems may include rods resting against rough or smooth walls and on rough ground. Bodies may be on an inclined plane. Consideration of jointed rods is not required. Questions involving toppling will not be set.

Topics	Guidance
2.6.5 Differential Equations	
Use differential equations in modelling in kinematics.	To include the use of first and second order differential equations.
Understand and use simple harmonic motion.	Candidates will be expected to set up the differential equation of motion, identify the period, amplitude and appropriate forms of solution. • $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\omega^2 x,$ • $T = \frac{2\pi}{\omega}, \; v^2 = \omega^2(a^2 - x^2)$ • $x = a\sin(\omega t),\, x = a\cos(\omega t)$ Candidates may quote formulae in problems unless the question specifically requires otherwise. Questions may involve light elastic strings or springs. Learners should: • understand the term damping, • know that a mathematical model for SHM may be refined to include damping, i.e. including a resistive term that varies with v, $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2k\frac{\mathrm{d}x}{\mathrm{d}t} + \omega^2 x = 0$$ Candidates should be able to interpret their solutions in cases where a system is

Topics	Guidance
	• Overdamped, $k^2 - \omega^2 > 0$ • Critically damped, $k^2 - \omega^2 = 0$ • Underdamped, $k^2 - \omega^2 < 0$ GeoGebra links: https://www.geogebra.org/m/ZzKbkvnz https://www.geogebra.org/m/yNbXCwcJ Angular S.H.M. is not included.

Notes on modelling
Learners should be able to • state any necessary modelling assumptions. For example, - objects/bodies can be modelled as particles (including for problems involving impact), - strings and springs will be light, - strings will be inextensible unless otherwise stated, - ‘rough’ (‘smooth’) implies that friction must be considered (ignored), - no external forces are present unless otherwise stated, e.g. air resistance can be ignored, - acceleration due to gravity, g, will be assumed to be constant, irrespective of height and vertical distance travelled. • understand that mechanical energy is lost when possible external forces are considered; • reflect on the significance of the value of e or a range of values of e; • identify possible limitations of any modelling assumptions made; • briefly describe the mathematical consequences if certain modelling requirements are not met; • suggest possible refinements to a chosen mathematical model. Attempts should be made to relate questions to real-life settings wherever possible. For example; sporting situations, bungee jumping, fairground rides and transport.

Assessment Objective 1 (AO1)

ASSESSMENT OBJECTIVES AO1: Use and apply standard techniques	Weighting	
	AS	A Level
Learners should be able to: • select and correctly carry out routine procedures; and • accurately recall facts, terminology and definitions	45% – 55%	45% – 55%

Assessment Objective 2 (AO2)

ASSESSMENT OBJECTIVES AO2: Reason, interpret and communicate mathematically	Weighting	
	AS	A Level
Learners should be able to: • construct rigorous mathematical arguments (including proofs); • make deductions and inference; • assess the validity of mathematical arguments; • explain their reasoning; and • use mathematical language and notation correctly.	20% – 30%	20% – 30%

Assessment Objective 3 (AO3)

ASSESSMENT OBJECTIVES AO3: Solve problems within mathematics and in other contexts	Weighting	
	AS	A Level
Learners should be able to: • translate problems in mathematical and non-mathematical contexts into mathematical processes; • interpret solutions to problems in their original context, and, where appropriate, evaluate their accuracy and limitations; • translate situations in context into mathematical models; • use mathematical models; and • evaluate the outcomes of modelling in context, recognise the limitations of models and, where appropriate, explain how to refine them.	**20% – 30%**	**20% – 30%**

Mathematical Problem Solving

Attributes of a problem solving task:

- Little or no scaffolding.
- Multiple representations, eg. sketch/diagram as well as calculations.
- Information is not given in mathematical form/language; or interpretation of results or evaluation of methods in a real world context.
- Variety of techniques that could be used.
- Solution requires understanding of the processes involved.
- Two or more mathematical processes required or different parts of mathematics to be brought together to reach a solution.

(*A Level Mathematics Working Group Report,* December 2015, Ofqual/15/5789)

AS Unit 1: Further Pure Mathematics A

- Further Vectors
  - (i) vector and Cartesian forms of the equation of a straight line in 3-D.
  - (ii) vector and Cartesian forms of the equation of a plane.
  - (iii) use scalar product to calculate the angle between two lines, two planes and between a line and a plane.
  - (iv) intersection of a line and a plane.
  - (v) perpendicular distance between two lines, from a point to a line and a point to a plane.

AS Unit 3: Further Mechanics A

- Circular Motion
  - (i) Motion in a horizontal circle where a particle is threaded on one string.
  - (ii) Motion in a horizontal circle where a particle is constrained by two strings, one of which is elastic.

- Circular Motion/Conservation of energy: Motion in a vertical circle where a particle is moving on the outside of a part circle.

A2 Unit 4: Further Pure Mathematics B

- Further Calculus (i) evaluate the mean value of a function.
- Differential Equations (i) use an integrating factor to solve differential equations of the form $\dfrac{dy}{dx} + P(x).y = Q(x)$

A2 Unit 6: Further Mechanics B

- Momentum and Impulse.

AS Unit 1: Straight Lines in 3-D and Planes

The vector equation of a straight line through $P(a, b, c)$ which has position vector

$\mathbf{p} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and is parallel to $\mathbf{q} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$ is

$$\mathbf{r} = \mathbf{p} + \lambda\mathbf{q}$$

$$\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} + \lambda(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})$$

or $$\mathbf{r} = (a + \lambda l)\mathbf{i} + (b + \lambda m)\mathbf{j} + (c + \lambda n)\mathbf{k}$$

Let a general point on $\mathbf{r}$ be (x, y, z)

therefore $$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (a + \lambda l)\mathbf{i} + (b + \lambda m)\mathbf{j} + (c + \lambda n)\mathbf{k}$$

$$x = a + \lambda l$$

$$y = b + \lambda m$$

$$z = c + \lambda n$$

therefore $$\lambda = \frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$$

So the Cartesian equation of a straight line through (a, b, c) and parallel to $l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$ is

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}.$$

The vector equation of a plane which contains the point A, with position vector $\mathbf{a}$, and is parallel to the vectors $\mathbf{b}$ and $\mathbf{c}$ is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

Given a vector $\boldsymbol{n}$ which is perpendicular to the plane, then

$$\mathbf{r}.\,\mathbf{n} = (\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}).\,\mathbf{n}$$

$$= \mathbf{a}.\mathbf{n} + \lambda\mathbf{b}.\mathbf{n} + \mu\mathbf{c}.\mathbf{n}$$

therefore $\mathbf{r}.\,\mathbf{n} = \mathbf{a}.\mathbf{n}$ (since $\mathbf{b}.\mathbf{n} = \mathbf{c}.\mathbf{n} = 0$)

or $\mathbf{r}.\,\mathbf{n} = d$ is the vector equation of the plane.

Let a general point in the plane be (x, y, z) and a vector perpendicular to the plane be $\mathbf{n}$ where $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then

$$\mathbf{r}.\,\mathbf{n} = d$$

$$(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = d$$

giving $ax + by + cz = d$ which is the Cartesian equation of the plane.

Examples

1. The line l passes through the point $P(3, 2, 5)$ and is perpendicular to the plane Π whose vector equation is

$$\mathbf{r}.(2\mathbf{i} - \mathbf{j} + 4\mathbf{k}) = 3$$

Find

a) the vector equation of the line l,
b) the position vector of the point where the line l meets the plane Π.
c) Hence find the perpendicular distance of P from Π.

Solution

a) $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + 4\mathbf{k})$

b) The equation of l: $\mathbf{r} = (3 + 2\lambda)\mathbf{i} + (2 - \lambda)\mathbf{j} + (5 + 4\lambda)\mathbf{k}$

Substitute into Π

$[(3 + 2\lambda)\mathbf{i} + (2 - \lambda)\mathbf{j} + (5 + 4\lambda)\mathbf{k}].(2\mathbf{i} - \mathbf{j} + 4\mathbf{k}) = 3$

$2(3 + 2\lambda) - 1(2 - \lambda) + 4(5 + 4\lambda) = 3$

$24 + 21\lambda = 3$

$\lambda = -1$

the position vector of the point $\mathbf{r} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$

c) Position vector of P is $\mathbf{p} = 3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$ and the position vector of Q, where the line meets the plane, is $\mathbf{q} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$

$\mathbf{PQ} = \mathbf{q} - \mathbf{p} = -2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$

$PQ = \sqrt{(-2)^2 + 1^2 + (-4)^2}$

$= \sqrt{21}$

2. The line L passes through the points $A(5, -1, 2)$ and $B(6, 3, -1)$.

a) (i) Find the vector equation of the line L.

(ii) Find the Cartesian form of the equation of L.

b) The plane Π has equation $2x + 3y - 4z = 51$.

(i) Find the coordinates of the point of intersection of L and Π.

(ii) Find the acute angle between L and Π.

Solution

a) (i) $\mathbf{AB} = \mathbf{b} - \mathbf{a} = (6\mathbf{i} + 3\mathbf{j} - \mathbf{k}) - (5\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

$= \mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$

equation of L : $\mathbf{r} = \mathbf{a} + \lambda\mathbf{AB}$

$$\mathbf{r} = 5\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} - 3\mathbf{k})$$

(ii) Equation of L : $\mathbf{r} = (5 + \lambda)\mathbf{i} + (-1 + 4\lambda)\mathbf{j} + (2 - 3\lambda)\mathbf{k}$

Let the point on L be $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

so $x = 5 + \lambda$

$y = -1 + 4\lambda$

$z = 2 - 3\lambda$

$$\lambda = x - 5 = \frac{y+1}{4} = \frac{2-z}{3}$$

Equation of L : $\dfrac{x-5}{1} = \dfrac{y+1}{4} = \dfrac{z-2}{-3}$

b) (i) The line intersects the plane $2x + 3y - 4z = 51$ where

$2(5 + \lambda) + 3(-1 + 4\lambda) - 4(2 - 3\lambda) = 51$

$-1 + 26\lambda = 51$

$\lambda = 2$

point of intersection = $7\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}$, therefore coordinates (7, 7, –4).

(ii) Equation of Π : $\mathbf{r}.(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) = 51$,

therefore, the direction of the normal to Π is $2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$

To find the angle θ between L and the normal to Π:

$(\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}).(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) = |\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}| \times |2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}| \times \cos\theta$

$$2 + 12 + 12 = \sqrt{26} \times \sqrt{29} \times \cos\theta$$

$\cos\theta = \dfrac{26}{\sqrt{26}.\sqrt{29}}$ therefore $\theta = 18.76^0$

so the angle between L and $\Pi = 90^0 - 18.76^0 = 71.24^0$.

3. The plane Π contains the three points $A(1, 1, -2)$, $B(3, -1, -4)$ and $C(2, 4, 9)$.

Find the vector equation of Π in the form $\mathbf{r.n} = d$.

Write down the Cartesian equation of Π.

Solution

Let $\mathbf{r.n} = 1$ where $\mathbf{n} = p\mathbf{i} + q\mathbf{j} + r\mathbf{k}$

For A $\quad p + q - 2r = 1 \quad (1)$

For B $\quad 3p - q - 4r = 1 \quad (2)$

For C $\quad 2p + 4q + 9r = 1 \quad (3)$

Using equations (1) and (2): $4p - 6r = 2$

Using equations (2) and (3): $14p - 7r = 5$

giving $\quad p = \frac{2}{7},\ r = -\frac{1}{7}$

and substituting into equation (1): $\quad q = \frac{3}{7}$

Therefore $\mathbf{r.n} = 1 \Rightarrow \mathbf{r}.(\frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} - \frac{1}{7}\mathbf{k}) = 1$

or $\quad \mathbf{r}.(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = 7$ (vector form)

and $\quad 2x + 3y - z = 7$ (Cartesian form)

4. Find the shortest distance from the point $P(5, -4, 1)$ to the plane Π which is given by the equation $\mathbf{r}.(6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = 6$

Solution

Using the formula given in the Formula Booklet:

$$D = \frac{|n_1\alpha + n_2\beta + n_3\chi - k|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$$, where ((α, β, χ) are the coordinates of the point and $n_1x + n_2y + n_3z = k$ is the equation of the plane.

Given $P(5, -4, 1)$ and equation of the plane: $6x - 2y + 3z = 6$

$$D = \frac{|6\times5 + (-2)\times(-4) + 3\times1 - 6|}{\sqrt{6^2 + (-2)^2 + 3^2}}$$

Therefore $D = \frac{35}{7} = 5$

5. Two skew lines l_1 and l_2 have equations

$$l_1 : \mathbf{r} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(-\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$$

$$l_2 : \mathbf{r} = 2\mathbf{i} - 4\mathbf{j} + \mu(3\mathbf{j} + 2\mathbf{k})$$

Respectively, where λ and μ are parameters.

a) Find a vector in the direction of the common perpendicular to l_1 and l_2.

b) Find the shortest distance between l_1 and l_2.

Solution

Using the formula given in the Formula Booklet:

$D = \dfrac{|(\mathbf{b}-\mathbf{a}).\mathbf{n}|}{|\mathbf{n}|}$, where **a** and **b** are position vectors of points on each line and **n** is a mutual perpendicular to both lines.

a) Let $\mathbf{n} = p\mathbf{i} + q\mathbf{j} + r\mathbf{k}$ be the vector which is perpendicular to both lines.

The direction vector for l_1 is $-\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and for l_2 is $3\mathbf{j} + 2\mathbf{k}$.

Therefore $(p\mathbf{i} + q\mathbf{j} + r\mathbf{k}).(-\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) = 0 \quad \Rightarrow \quad -p + 3q + 4r = 0$

and $(p\mathbf{i} + q\mathbf{j} + r\mathbf{k}).(3\mathbf{j} + 2\mathbf{k}) = 0 \quad \Rightarrow \quad 3q + 2r = 0$

If $r = t$ then $q = -\frac{2}{3}t$ and $p = 2t$.

So $\mathbf{n} = 2t\mathbf{i} - \frac{2}{3}t\,\mathbf{j} + t\mathbf{k}$, putting $t = 3$, for example, gives $\mathbf{n} = 6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

b) Let point on l_1 be $\mathbf{a} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ (when $\lambda = 0$)

and the point on l_2 be $\mathbf{b} = 2\mathbf{i} - 4\mathbf{j}$ (when $\mu = 0$)

Therefore $D = \dfrac{|(\mathbf{b}-\mathbf{a}).\mathbf{n}|}{|\mathbf{n}|}$ where $(\mathbf{b} - \mathbf{a}) = (2\mathbf{i} - 4\mathbf{j}) - (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$

and $(\mathbf{b} - \mathbf{a}).\mathbf{n} = (\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}).(6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = 6 + 6 - 6 = 6$

$|\mathbf{n}| = \sqrt{6^2 + (-2)^2 + 3^2} = 7$

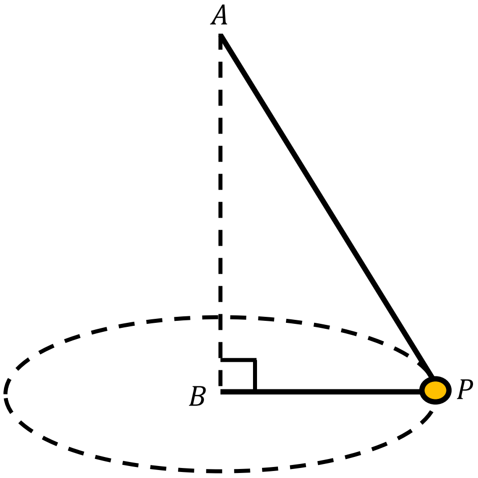
therefore $D = \dfrac{6}{7}$

AS Unit 3: Circular Motion – Motion in a horizontal circle where a particle is threaded on one string.

Example

A smooth bead P of mass m kg is **threaded** on a light inextensible string of length $2 \cdot 5$ m. One end of the string is fixed to a point A and the other end is fixed to a point B, where A is $1 \cdot 5$ m vertically above B.

The diagram below shows the bead P describing horizontal circles having centre B with uniform angular velocity ω radians per second.

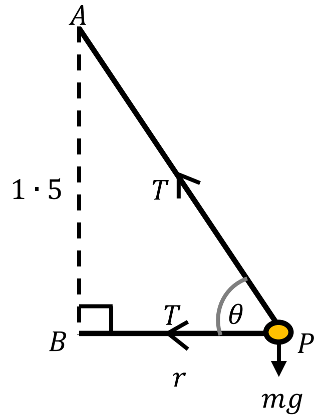


(a) Show that the radius of the circle is $0 \cdot 8$ m.

(b) Find the value of ω.

(c) State what would happen to the value of ω if the mass of the bead were increased.

Solution



(a) Using Pythagoras

$$(2 \cdot 5 - r)^2 = 1 \cdot 5^2 + r^2$$
$$6 \cdot 25 - 5r + r^2 = 2 \cdot 25 + r^2$$
$$r = \frac{6 \cdot 25 - 2 \cdot 25}{5} = 0 \cdot 8 \text{ m}$$

(b) Since the particle is free to move along the string, the tension, T, must be constant throughout.

Resolving vertically,

$$T\sin\theta = mg$$
$$T = \frac{mg}{\sin\theta} = mg \times \frac{1{\cdot}7}{1{\cdot}5}$$

Using N2L towards B,

$$T + T\cos\theta = ma$$
$$T(1+\cos\theta) = mr\omega^2 \qquad (a = r\omega^2)$$
$$\omega^2 = \frac{T(1+\cos\theta)}{mr} = mg \times \frac{1{\cdot}7}{1{\cdot}5} \times \frac{\left(1+\frac{0{\cdot}8}{1{\cdot}7}\right)}{0{\cdot}8m} = 20 \cdot 416\ldots$$

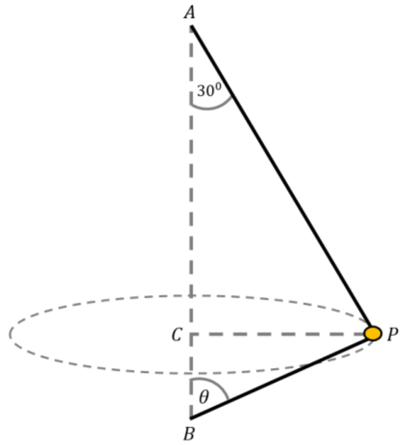
$$\omega = \sqrt{20 \cdot 416\ldots} = 4 \cdot 52 \text{ rad s}^{-1}$$

(c) The value of ω is independent of m and so ω would remain the same.

AS Unit 3: Circular Motion – Motion in a horizontal circle where a particle is constrained by two strings, one of which is elastic.

Example

The diagram below shows a particle P, of mass 2 kg, attached by means of one light elastic string fixed at point A, together with one light inextensible string fixed at point B. The particle P describes a horizonal circle with centre C, radius $r\text{ m}$ and constant speed $v\text{ m s}^{-1}$. Point A is vertically above B with the elastic string, AP, being stretched but maintaining a constant length and BP is taut. AP makes an angle of 30^0 with the vertical and BP is inclined at an angle θ to the vertical where $\cos\theta = \frac{5}{13}$.



The tension in the elastic string AP is $8\sqrt{3}\,g\text{ N}$.

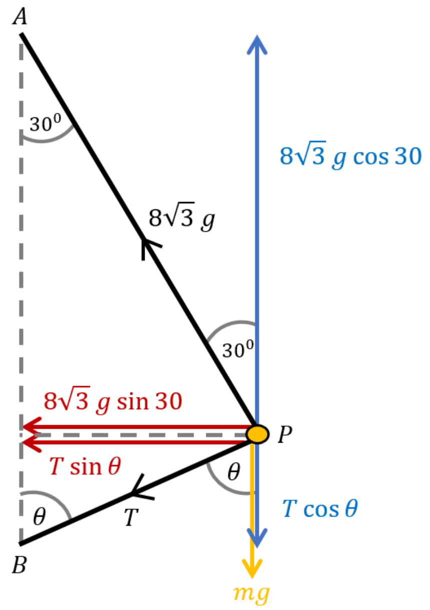
(a) Calculate the tension in the string BP, giving your answer in terms of g.

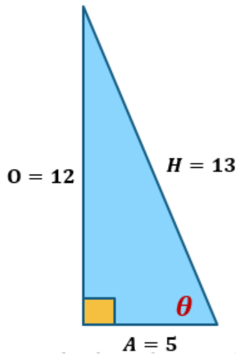
(b) Given that the original length of the elastic string is 3 m and $v^2 = 33g$,

(i) find the value of r and hence show that the $AP = (6 - \sqrt{3})\text{ m}$,

(ii) determine the value of the modulus of elasticity, λ, of the elastic string AP, giving your answer correct to one decimal place.

Solution





$$\cos\theta = \frac{5}{13}$$

$$\sin\theta = \frac{12}{13}$$

(a) Resolving vertically,

$$8\sqrt{3}\,g\ \cos 30 = T\ \cos\theta + 2g$$

$$8\sqrt{3}\,g\left(\frac{\sqrt{3}}{2}\right) = T\left(\frac{5}{13}\right) + 2g$$

$$12g = \frac{5T}{13} + 2g$$

$$\therefore \qquad T = 26g\text{ N}$$

(b) (i) Using N2L towards C, with $v^2 = 33g$,

$$8\sqrt{3}\,g\ \sin 30 + 26g\sin\theta = \frac{2v^2}{r} \qquad \left(a = \frac{v^2}{r}\right)$$

$$8\sqrt{3}\,g\left(\frac{1}{2}\right) + 26g\left(\frac{12}{13}\right) = \frac{66g}{r} \qquad (\div\, g)$$

$$4\sqrt{3} + 24 = \frac{66}{r}$$

$$r = \frac{66}{4\sqrt{3}+24} = \frac{6-\sqrt{3}}{2}$$

$$\sin 30 = \frac{r}{AP} \qquad \therefore \qquad AP = \frac{r}{\sin 30} = 2r = 2\left(\frac{6-\sqrt{3}}{2}\right) = 6-\sqrt{3}$$

(ii) Using $\text{T} = \frac{\lambda x}{l}$ with $T = 8\sqrt{3}\,g$, $l = 3$ and $x = (6-\sqrt{3}) - 3 = 3-\sqrt{3}$,

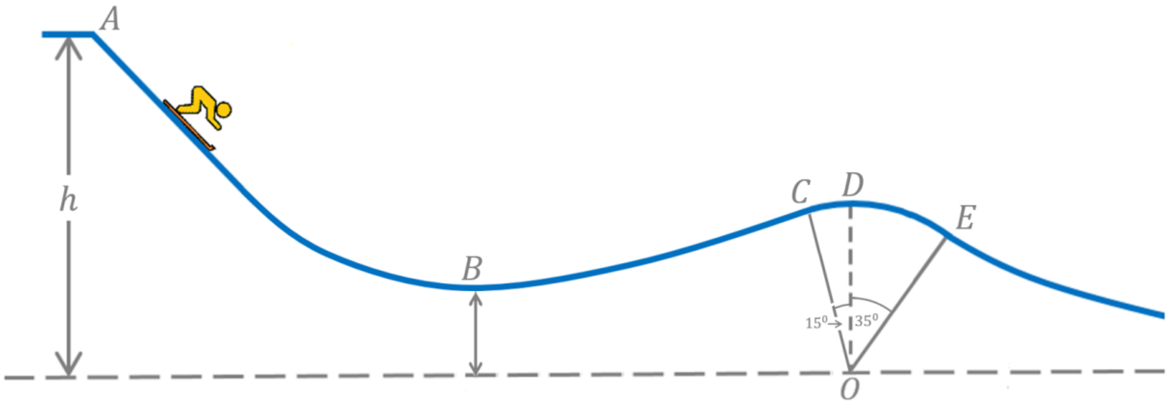
$$8\sqrt{3}\,g = \frac{\lambda(3-\sqrt{3})}{3}$$

$$\lambda = \frac{24\sqrt{3}\,g}{3-\sqrt{3}} = (12+12\sqrt{3})g = 321\cdot 3\text{ N}$$

AS Unit 3: Circular Motion/Conservation of energy: Motion in a vertical circle where a particle is moving on the outside surface of a part circle.

Example

The diagram shows a vertical cross section of a ski slope between the points A and E. The starting point is at A and a lower part of the slope CDE may be modelled as an arc of a circle of radius 4 m with centre O and D vertically above O. Angles $C\hat{O}D$ and $D\hat{O}E$ are 15^0 and 35^0 respectively. Point A is $h\text{ m}$ vertically above O, where $h > 4$. B is the lowest point between A and E and is $2 \cdot 5\text{ m}$ vertically above O. A skier, who is to be modelled as particle P of mass 80 kg, sets off from A with speed is 2 m s^{-1}. The speed of P when OP makes an angle θ with the upward vertical is $v\text{ m s}^{-1}$.



(a) Find, in terms of h, the greatest speed attained by the skier during the journey from A to E.

(b) For the journey from C to E,
  (i) show that $v^2 = 4 + 2g(h - 4\cos\theta)$,
  (ii) find, in terms of θ, g and h, the reaction of the slope on the skier,
  (iii) determine the greatest possible value of h such that the skier will remain in contact with the slope.

(c) In addition to the assumption given in the question, write down one further assumption that you have made in your solution to this problem and briefly explain how this assumption affects your conclusion in (b)(iii).

Solution

(a) Greatest speed will occur at point B, the lowest point in the journey.
Using conservation of energy,

$$\text{KE at } B + \text{PE at } B = \text{PE at } A + \text{KE at } A$$

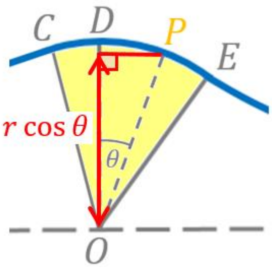
$$\tfrac{1}{2}mv_{max}^2 + mg(2 \cdot 5) = mgh + \tfrac{1}{2}m(2)^2 \qquad (\div m)$$

$$v_{max}^2 = 2gh + 4 - 5g$$

$$v_{max} = \sqrt{9 \cdot 6h - 45}$$

(b) Suppose that P is a general point on the arc CE.

(i) Using conservation of energy, with $u = 2$ and $r = 4$,



$$\text{PE at } A + \text{KE at } A = \text{PE at } P + \text{KE at } P$$

$$mgh + \frac{1}{2}m(2)^2 = mgr\cos\theta + \frac{1}{2}mv^2 \qquad (\div m)$$

$$gh + \tfrac{1}{2}(2)^2 = 4g\cos\theta + \tfrac{1}{2}v^2$$

Rearranging to make v^2 the subject gives,

$$v^2 = 2gh + 4 - 8g\cos\theta,$$

$$v^2 = 4 + 2g(h - 4\cos\theta) \qquad \text{as required.}$$

(ii) Suppose that the reaction is given by R. Then, by using N2L towards O with $r = 4$ and $m = 80$,

$$mg\cos\theta - R = \frac{mv^2}{r}$$

$$80g\cos\theta - R = \frac{mv^2}{r} = \frac{80v^2}{4} = 20v^2$$

Rearranging to make R the subject gives,

$$R = 80g\cos\theta - 20v^2$$

$$R = 80g\cos\theta - 20(4 + 2g(h - 4\cos\theta)) \qquad \text{(using } v^2 \text{ from part (i))}$$

$$R = 80g\cos\theta - 40gh - 80 + 160g\cos\theta$$

$$R = 240g\cos\theta - 40gh - 80$$

(iii) The skier will remain in contact with the slope if $R > 0$. Since 35^0 is the maximum possible value of θ (either side of the vertical) and $\cos\theta$ is decreasing, we can deduce that $\cos\theta \geq \cos 35^0$.
Therefore, we need

$$240g\cos\theta - 40gh - 80 \geq 240g\cos 35^0 - 40gh - 80 > 0$$

$$240g\cos 35^0 > 40gh + 80$$

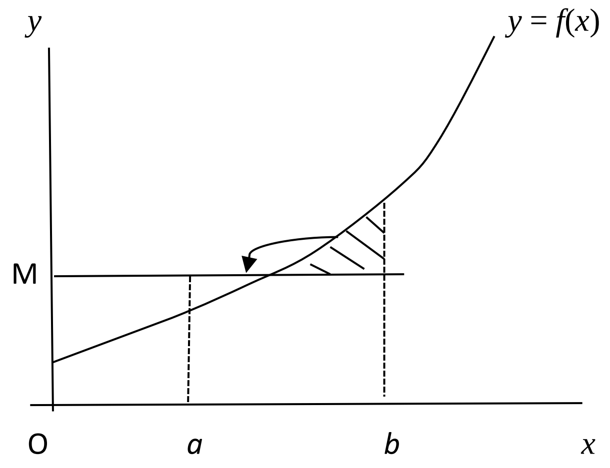
$$h < \frac{240g\cos 35^0 - 80}{40g} = 4\cdot 71 \text{ m} \qquad \text{(2 dp)}$$

Therefore, skier will remain in contact with the slope for $h < 4\cdot 71$

(c) In addition to the assumption given in the question, we have assumed that there are no external forces acting on the particle such as friction between the skiis and the slope or air resistance. Additional forces acting against the skier would result in work being done to decrease the energy in the system. As a result,

- $v = 0$ is a possibility and so the skier would stop at some point,
- the upper limit for h could be greater than $4\cdot 71$.

A2 Unit 4: Further Calculus - Evaluate the Mean value of a function.



Let M be an estimate of the mean height of the function $y = f(x)$ from $x = a$ to $x = b$ such that the part of the enclosed region cut off would fill the space below. In other words, the area of the figure between $y = f(x)$, the x-axis, $x = a$ and $x = b$ is equal to the area of the rectangle produced.

$$\text{Area} = M(b - a)$$

$$M = \frac{1}{b-a} \times \text{Area}$$

$$\text{Mean value } M = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Examples

1. Find the mean value of $y = 3x^2 + 4x + 1$ between $x = -1$ and $x = 2$.

Solution

$$\text{Mean value} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2-(-1)}\int_{-1}^{2}(3x^2+4x+1)\,dx$$

$$= \frac{1}{3}\left[x^3 + 2x^2 + x\right]_{-1}^{2}$$

$$= \frac{1}{3}\left[(8+8+2)-(-1+2-1)\right]$$

$$= 6$$

2. Find the mean value of $y = 3\sin 5x + 2\cos 3x$ between $x = 0$ and $x = \pi$.

Solution

Mean value $= \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,\mathrm{d}x = \dfrac{1}{\pi - 0}\int_0^\pi (3\sin 5x + 2\cos 3x)\,\mathrm{d}x$

$$= \frac{1}{\pi}\left[\frac{-3\cos 5x}{5} + \frac{2\sin 3x}{3}\right]_0^\pi$$

$$= \frac{1}{\pi}\left[\left(\frac{3}{5} + 0\right) - \left(\frac{-3}{5} + 0\right)\right]$$

$$= \frac{6}{5\pi}$$

A2 Unit 4: Differential Equations - Using an Integrating Factor to solve differential equations of the form $\dfrac{\mathbf{d}y}{\mathbf{d}x} + P(x).y = Q(x)$

Step 1: Find the Integrating Factor: $e^{\int P(x)\mathrm{d}x}$.

Step 2: Multiply both sides of the equation by the Integrating Factor.

Step 3: Integrate both sides: y × Integrating Factor = $\int Q(x) \times$ Integrating Factor $\mathrm{d}x$

Examples

1. Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\cot x = 2\cos x$$

a) Find an integrating factor for this differential equation.

b) Solve the differential equation given that y = 0 when x = $\frac{\pi}{2}$.

Solution

a) Integrating Factor = $e^{\int \cot x\mathrm{d}x}$

Since $\int \cot x\mathrm{d}x = \int \frac{\cos x}{\sin x}\mathrm{d}x$ = $\ln(\sin x)$

Integrating Factor = $e^{\ln(\sin x)} = \sin x$

b) Multiplying both sides of the differential equation by the integrating factor:

$$\sin x\frac{\mathrm{d}y}{\mathrm{d}x} + y\cot x\sin x = 2\cos x\sin x$$

Integrating: $y \times \sin x = \int \sin 2x\mathrm{d}x$

$$y\sin x = \frac{-\cos 2x}{2} + c$$

when $y = 0$, $x = \frac{\pi}{2}$

$$0 = \frac{1}{2} + c, \text{ therefore } c = -\frac{1}{2}$$

Solution is $y\sin x = \dfrac{-\cos 2x}{2} - \dfrac{1}{2}$ or

or $y\sin x = -\dfrac{1}{2}(\cos 2x + 1)$

2. Consider the differential equation

$$\frac{dy}{dx} + y\tan x = \tan x \sec x$$

a) Find an integrating factor for this differential equation.

b) Solve the differential equation given that $y = 5$ when $x = \frac{\pi}{3}$.

Solution

a) Integrating Factor = $e^{\int \tan x dx}$

Since $\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\ln(\cos x) = \ln(\sec x)$

Integrating Factor = $e^{\ln(\sec x)} = \sec x$

b) Multiplying both sides of the differential equation by the integrating factor:

$$\sec x \frac{dy}{dx} + y\tan x \sec x = \tan x \sec^2 x$$

Integrating: $y \times \sec x = \int \tan x \sec^2 x dx$

$$y \sec x = \frac{1}{2}\tan^2 x + c$$

when $y = 5$, $x = \frac{\pi}{3}$,

$5 \times 2 = \frac{1}{2} \times 3 + c$ therefore $c = \frac{17}{2}$

$$y \sec x = \frac{1}{2}\tan^2 x + \frac{17}{2}$$

$$y = \frac{1}{2}\cos x\,(\tan^2 x + 17)$$

3. Solve the differential equation

$$x\frac{dy}{dx} + y = 5x^3$$

given that $y = 7$ when $x = 2$.

<u>Solution</u>

First of all, divide both sides of the equation by x.

$$\frac{dy}{dx} + \frac{1}{x}y = 5x^2$$

Integrating Factor = $e^{\int \frac{1}{x}dx}$ = $e^{\ln x}$ = x

Multiplying both sides of the differential equation by the integrating factor and integrating:

$yx = \int 5x^3 dx$

$yx = \frac{5}{4}x^4 + c$

when $x = 2$ $y = 7$: 14 = 20 + c

c = – 6

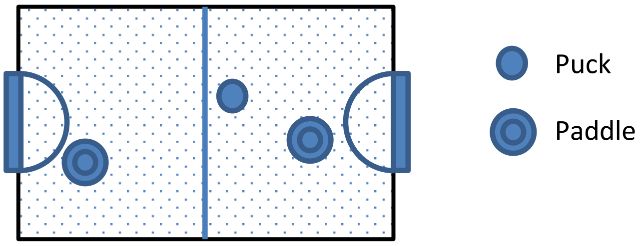
which gives $yx = \frac{5}{4}x^4 - 6$

or $y = \frac{5}{4}x^3 - \frac{6}{x}$

A2 Unit 6: Momentum and Impulse

Example

Lowri and Dylan are playing a game of air-hockey. The table, shown in the diagram below, is modelled as the horizontal x-y plane with the point O as the origin and unit vectors parallel to the x-axis and the y-axis denoted by $\mathbf{i}$ and $\mathbf{j}$ respectively. The puck has mass m kg and a paddle has mass $2m$ kg.



During play, the puck is moving on the table with velocity $(44\mathbf{i} + 15\mathbf{j})$ ms^{-1}. Lowri then moves the paddle with velocity $(-16\mathbf{i} + 4\mathbf{j})$ ms^{-1} so that it collides with the puck. On impact their line of centres is in the direction $\mathbf{i}$.

The coefficient of restitution for all collisions on the table is $\frac{4}{5}$.

(a) Show that the velocity of the puck after the collision is $(-28\mathbf{i} + 15\mathbf{j})$ ms^{-1}.

After the collision between the puck and the paddle, the puck collides with the table wall which is parallel to the vector $\mathbf{i}$.

(b) Calculate the velocity of the puck after impact with the table wall.

(c) Determine the impulse exerted by the table wall on the puck. Comment on your answer.

(d) Find the percentage loss in kinetic energy caused by the impact between the table wall and the puck. Give your answer to the nearest whole number.

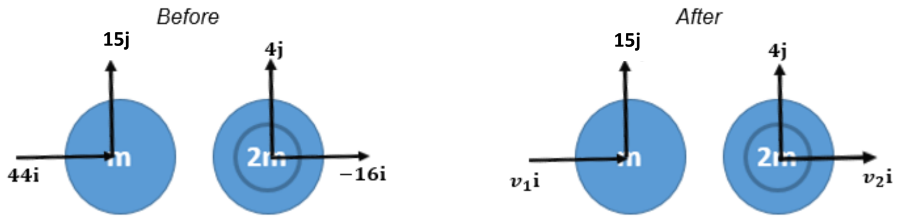
For Lowri to score a point, the puck must cross the goal line defined by the position vector $\mathbf{r} = k\mathbf{j}$, where $2 < k < 6$.

(e) Given that the puck has position vector $(7\mathbf{i} + 8\mathbf{j})$ m at the instant it collides with the table wall and that Dylan's reaction time is slightly greater than 0.25 s, determine if Lowri will score a point.

(f) State two modelling assumptions that you have made in your solutions. For one of your assumptions, explain why it may be unreasonable.

Solution

(a)



Velocities perpendicular to line of centres remain unchanged, i.e. $15\mathbf{j}$ and $4\mathbf{j}$.

Conservation of momentum parallel to **i** gives

$$44m - 16(2m) = v_1 m + 2m v_2$$

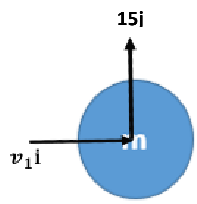
Restitution equation gives

$$v_2 - v_1 = -\tfrac{4}{5}(-16 - 44),$$

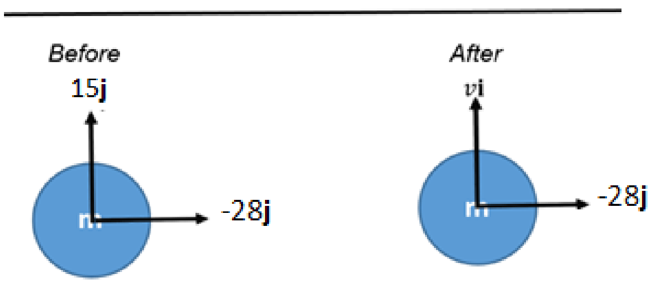
Solving simultaneously,

$$v_1 + 2v_2 = 12$$
$$v_2 - v_1 = 48$$
$$v_1 = -28 \qquad (\text{and } v_2 = 20)$$

Therefore, $\mathbf{v}_{puck} = -28\mathbf{i} + 15\mathbf{j}$



(b)



Velocity parallel to the wall remains unchanged, i.e. $-28\mathbf{i}$

Restitution equation gives $\quad v = -\tfrac{4}{5}(15) = -12.$

Therefore, $\quad \mathbf{v} = -28\mathbf{i} - 12\mathbf{j}$

(c) Impulse, $\mathbf{I}$ = change in momentum

$$\mathbf{I} = m(-28\mathbf{i} - 12\mathbf{j}) - \mathrm{m}(-28\mathbf{i} + 15\mathbf{j})$$

$$\mathbf{I} = -27m\mathbf{j} \text{ (Ns)}$$

Impulse is perpendicular to table wall since we have no **i** component.

(d) Loss in KE = $\frac{1}{2}m(784 + 225) - \frac{1}{2}m(784 + 144)$

Loss in KE = $\frac{81m}{2}$ (J)

Percentage in KE = $\frac{81m/_2}{\frac{1}{2}m(784+225)} \times 100 = 8\%$

(e) $\mathbf{r}_{puck} = \mathbf{r_o} + \mathbf{v}t = 7\mathbf{i} + 8\mathbf{j} + (-28\mathbf{i} - 12\mathbf{j})t$

Let $t = 0 \cdot 25$ to get $\mathbf{r}_{puck} = 5\mathbf{j}$.

$5\mathbf{j}$ lies on the goal line since it is between $2\mathbf{j}$ and $6\mathbf{j}$. Therefore Lowri scores a point since Dylan does not react in time.

(f) Two valid assumptions, eg.

- Puck and paddle treated as particles
- Table is smooth (and horizontal)
- Uniform shapes with equal radii
- Holding the paddle will not have an effect

Any valid explanation as to why assumption may be unreasonable, eg.

- Holding paddle will certainly add extra mass and it is likely that there will be acceleration.
- Table is not perfectly smooth so constant velocity is unreasonable.